

The Least Eigenvalue of the Input Coefficients Matrix

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Abstract- It is well known that the input coefficients matrix, called direct consumption coefficients matrix in China, plays an important role in Input-Output Analysis. The matrix with nonnegative entries has n eigenvalues, and the first two eigenvalues have been studied and been given their economic meanings in some articles published in the journal Economic Systems Research. In details, the dominant Frobenius eigenvalue determines the turnpike growth rate, and the subdominant eigenvalue determines the speed of a convergence. But few articles are given to research the economic meaning of the least eigenvalue in terms of absolute magnitude. In this paper, we discuss it based on linear system theory and finally give its economic meaning in some dynamic models. In an empirical analysis for China between 1987 and 2007, results are calculated for the least eigenvalues in modulus of the direct consumption coefficients matrices.

Index Terms- Matrix, eigenvalue, linear system theory

I. INTRODUCTION

In Input-Output Analysis, input coefficients can reflect the information of production technology. So the input coefficients are called technical coefficients, and are called direct consumption coefficients in China. As a whole, the input coefficients matrix can reflect the economic system's technological level.

From a mathematical point of view, eigenvalues and eigenvectors, or called characteristic values and characteristic vectors respectively, are two important conceptions for a square matrix. Therefore, for an input coefficients matrix with specific economic meaning, its eigenvalues and eigenvectors must have unique economic meanings. In this paper, we only consider its eigenvalues. For a $n \times n$ matrix, there are n eigenvalues in real number field, and for the two largest eigenvalues in terms of absolute magnitude, some economists have provided their economic meanings, i.e. the largest (or called dominant Frobenius eigenvalue) determines the turnpike growth rate, and the second largest eigenvalue determines the speed of a convergence. But few researchers study the economic meaning of the least eigenvalue in terms of absolute magnitude. In this paper, we study the least eigenvalue of the input coefficients matrix based on linear system theory.

We know that linear system theory is widely used in the research field of control and system theory, in which the mainly research system is dynamics system. Dynamics system can be denoted by a group of differential equations, or difference equations. When the mathematical equations have the linear characteristics, the corresponding system is defined as linear systems which is the simplest and is widely researched above all. It is well known that the

basic characteristic of linear systems is that it satisfies linear operation, namely, if dynamics system can be represented by mathematical function L , the following equation is correct with two input variables u_1 and u_2 ,

$$L(c_1u_1 + c_2u_2) = c_1L(u_1) + c_2L(u_2)$$

where c_1 and c_2 are two arbitrary constants.

In actual economic society, John Maynard Keynes, a notable British economist thought national income is made up of consumption and investment. Therefore, in a country, the process of input, production and consumption can be considered as a complex system. Actually, of course, the economic system is not linear systems. But in a limited period and range, the relations of major variables in the economic system can be written appreciatively using linear equation. Moreover, the difference between actual economic system and the idealization linear economic system can be largely neglected for our researching problem.

In this paper, the first problem be resolved is how to represent economic system by linear mathematical equations, and Input-Output Analysis in economics gives an available approach. Then we provide some dynamic models involved consumption, input and output vectors. By linear system theory, we analyze these models and finally give the economic meaning of the least eigenvalue. Meanwhile, we calculate some countries such as China, America, Japan and Britain, these countries' empirical data on least eigenvalues.

II. METHODOLOGY

2.1 Notations, definition and theorem of linear system theory

For a dynamic system, let x_1, x_2, \dots, x_n be state variables group describing system action, u_1, u_2, \dots, u_p and y_1, y_2, \dots, y_q be input and output variables groups, respectively. The linear discrete system with constant coefficients can be written as follows:

$$\begin{cases} X(k+1) = GX(k) + HU(k), & k = 0, 1, 2, \dots \\ Y(k) = CX(k) + DU(k) \end{cases} \quad (1)$$

where $X(k), U(k), Y(k)$ are vectors with components $x_i(k), u_i(k), y_i(k)$, respectively, and constant matrices G, H, C, D are consistent with the system. Among the four matrices, matrix G , called coefficients matrix, determines the property of the dynamic system.

Definition When $k \rightarrow \infty$, state vector variable $X(k)$ tends to the origin in state space, then the system is termed stable gradually.

Lemma For linear system (1), input variables group $U(k) = 0$, then the system is stable gradually if and only if all the eigenvalues' absolute magnitude of coefficient matrix G are less than 1.

2.2 Hua's dynamic model with zero consumption

Suppose the national economy system in a country will be divided into n product sectors, and $n \times 1$ column vectors

$$X(t)^T = [x_1(t), x_2(t), \dots, x_n(t)]$$

be the t year's total gross output vector,

$$C(t)^T = [c_1(t), c_2(t), \dots, c_n(t)]$$

be the t year's consumption vector, and

$$Y(t-1)^T = [y_1(t-1), y_2(t-1), \dots, y_n(t-1)]$$

be the t year's input vector. The relation of the three vector variables is

$$Y(t) = X(t) - C(t).$$

In order to discuss models clearly, consider a special case, i.e. zero consumption vector, which means the entire products will be put in producing process. Suppose the input coefficient matrix A is inverse, nonnegative and irreducible, and its nonzero eigenvalues are $\lambda_1, \lambda_2, \dots, \lambda_n$. Among these eigenvalues, let λ_1 denotes the **Dominant Frobenius Eigenvalue**, and λ_n be the least eigenvalue in terms of absolute magnitude. According to characters of matrix polynomial on eigenvalue, for matrix A^{-1} , its eigenvalues are the reciprocal of λ_i , namely λ_i^{-1} , $i = 1, 2, \dots, n$.

The Hua's dynamic model with zero consumption vector can be written as follows

$$AX(t+1) = X(t) \tag{2}$$

multiply by A^{-1} on left sides above equation, this yields

$$X(t+1) = A^{-1}X(t) \tag{3}$$

This can be considered as a state equation in linear discrete system with constant coefficients and zero input. Because all absolute magnitude of the eigenvalues of input coefficients matrix A are less than 1, for linear economy system (3), the eigenvalues of the matrix A^{-1} are bigger than 1. By **Lemma** and **Definition** mentioned above, we have

Theorem 1 Linear economy system (3) is not stable gradually, i.e. $X(t)$ does not tend to zero when $t \rightarrow \infty$.

The above theorem illustrates, for a zero consumption economy system, it never dies out. In other words, the economy system is filling of strength and energy to grow.

For linear economy system (3), after have been finished several papers, **Luogeng Hua**, a famous mathematician in China finally got a conclusion named **Positive Eigenvector Method**, which means the necessary and sufficient condition of economic growth is that it should be linear correlation between the original input vector and the right positive eigenvector of the coefficient matrix A . Otherwise, several years later, some components of output vector will be negative. In terms of the proportion of the right positive eigenvector as input structure, relation of output and input vectors is

$$X(t+1) = \lambda_1^{-1}X(t)$$

where λ_1^{-1} is termed growth coefficient.

2.3 General dynamic model

Of course, that consumption vector is zero is impossible. The general dynamic model of (3) with consumption vector $C(t)$ can be written as follows:

$$\begin{cases} AX(t+1) + C(t) = X(t) \\ Y(t) = X(t) - C(t) \end{cases} \quad t = 1, 2, \dots \tag{4}$$

namely,

$$\begin{cases} X(t+1) = A^{-1}X(t) - A^{-1}C(t) \\ Y(t) = X(t) - C(t) \end{cases} \quad t = 1, 2, \dots \tag{5}$$

In difference equations (5), from a system theoretical point of view, it can be considered as linear discrete system with constant coefficients,

and total gross output vector $X(t)$ in economics is state variables group in linear system theory, input vector $Y(t)$ is input variables group, and consumption vector $C(t)$ is output variables group.

For linear system (5), we only consider a special simple condition, i.e., $Y(t) = \alpha_t X(t)$,

which means a part of t year's output $\alpha_t X(t)$ as t+1 year's input, and the left part as consumption. So, the state equation of linear economic system (5) becomes

$$X(t+1) = \alpha_t A^{-1} X(t) \tag{6}$$

where coefficient α_t is termed **gross input coefficient** with property $0 \leq \alpha_t < 1$.

2.4 The economic meaning of the least eigenvalue

Theorem 2 For linear discrete economy system (6), coefficient matrix A is nonnegative, irreducible and inverse, its eigenvalues are $\lambda_1, \lambda_2, \dots, \lambda_n$ numbered in a decreasing order by their absolute magnitude, and its **Frobenius positive eigenvector** is denoted by U , and gross input coefficient is α_t . Then

(1) when $\alpha_t < |\lambda_n|$, the system will perish asymptotically.

(2) when $\alpha_t \geq |\lambda_n|$, the system is active.

(3) when $\alpha_t > \lambda_1$, and it is linear correlation between the first year's input $Y(0)$

and U , the linear system is growing year and year.

Proof: According to properties of the matrix polynomials on eigenvalue, the eigenvalues of matrix $\alpha_t A^{-1}$ are

$$\alpha_t \lambda_1^{-1}, \alpha_t \lambda_2^{-1}, \dots, \alpha_t \lambda_n^{-1},$$

and their absolute magnitudes relationship is

$$\alpha_t \lambda_1^{-1} \leq \alpha_t |\lambda_2|^{-1} \leq \dots \leq \alpha_t |\lambda_n|^{-1}.$$

When $\alpha_t < |\lambda_n|$, all eigenvalues' absolute magnitudes of the matrix $\alpha_t A^{-1}$ are less than 1. Therefore, by **Lemma**, the system will perish asymptotically. So the conclusion (1) holds.

When $|\lambda_n| \leq \alpha_t$, the system is active, which means some sector will die out but some sector do not. Therefore, the conclusion (2) is correct.

For model (6), the **Positive Eigenvector Method** is still valid. Therefore conclusion (3) is true, and growth rate is $(\alpha_t - \lambda_1) \lambda^{-1}$.

To conclude this major section: based on Input-Output Analysis, we can describe economy system as a general dynamic model, or as a linear and discrete system from a linear system theoretical point of view. By linear system theory, we get the conclusions on the eigenvalues of input coefficients matrix after analyzing these dynamic models.

For an economy system, if all the output be put into next year's production process, i.e. zero consumption, the economy system is filling with strength and energy (But it does not mean that the economy system grows for ever because the input structure plays an important and key role). If one part of the total gross output as input be

put into next year's production processing and the left part as consumption, that whether or not the economy grows depends on the magnitude of input proportion in total gross output. Therefore, the least eigenvalue in terms of absolute magnitude and the largest eigenvalue, which is a positive and real number if the input coefficients matrix is irreducible, are two critical points. In detail, if the input part in total gross output is smaller than the least eigenvalue, the economy must die out; the necessary condition for the economy growth is that the input proportion in total gross output must be bigger than the largest eigenvalue.

III. EMPIRICAL DATA ON EIGENVALUES

Using software *Mathematica*, we can calculate the eigenvalues of four countries including China, America, Japan and Britain, and exhibit in the following tables.

Note that the economy system includes three industries, i.e. primary industry, secondary industry, and tertiary industry, i.e. the input coefficients matrices of these countries are 3x3 square matrices. China's data on input coefficients come from websites of China Input-Output Association, National Bureau of Statistics of China, and China Research Center of Tertiary Industry, Sun Yat-sen University.

Tab. 1 China's input coefficients

Year	Input coefficients		
1987 China	0.1472	0.0848	0.0240
	0.1314	0.4972	0.2438
1990 China	0.1788	0.0955	0.0226
	0.1277	0.5208	0.2587
	0.0363	0.0812	0.1463

1992 China	0.1393	0.0694	0.0147
	0.1568	0.5112	0.2775
	0.0597	0.1326	0.1992
1995 China	0.1723	0.0675	0.0148
	0.1728	0.5406	0.2602
	0.0572	0.1047	0.1736

Tab. 2 Other countries' input coefficients

Country	Input coefficients		
1967 America	0.2937	0.0438	0.0051
	0.1492	0.3865	0.0959
	0.1540	0.1436	0.2076
1975 Japan	0.1069	0.0538	0.0051
	0.2031	0.4806	0.1692
	0.0774	0.1349	0.1733
1980 Japan	0.1249	0.0401	0.0054
	0.2431	0.5066	0.1764
	0.0886	0.1343	0.1852
1963 Britain	0.1425	0.0147	0.0008
	0.3020	0.3688	0.1032
	0.1214	0.0964	0.0861

Tab. 3 China's eigenvalues

Year	Eigenvalues
	0.576301
1987	Mod(0.104899+0.00413426i)=0.10498
	Mod(0.104899 - 0.00413426i)=0.104817
	0.601937
1990	0.138758
	0.105205
	0.626991
1992	Mod(0.111354+0.0140523i)=0.112237
	Mod(0.111354 - 0.0140523i)=0.110464
	0.631842
1995	Mod(0.127329 +0.00591417i)=0.127466
	Mod(0.127329 - 0.00591417i)=0.127192

Tab. 4 Other countries' eigenvalues

Year	Eigenvalues
1967 America	0.485683
	0.246806
	0.155311
1975 Japan	0.567143
	0.109631
	0.084026
1980 Japan	0.590464
	0.121415
	0.104821
1963 Britain	0.417321
	0.124188
	0.055892

Comparing least eigenvalues of different countries and different years in above tables, we find the smallest is 0.055892 of Britain in 1963, and the largest is 0.155311 of America in 1967. China's least eigenvalues from 1987 to 1995 are approximate 0.11, and are becoming bigger and bigger.

By the conclusions on eigenvalues we get in section 3, America in 1967 should input more proportion in total gross output in order to ensure economy no depression. In detail, if the input part in total gross output is less than 15.53%, then the America's economy will be getting decrease. And for China, the proportion is about 12%, and for Britain, it is about 5.59%.

On the other hand, we look at the largest eigenvalues. Because the necessary condition for an economy growth is that the input part in total gross output is bigger than the largest eigenvalue, we find Britain in 1963 the largest eigenvalue is 0.417321 and America in 1967 is 0.485683. These data are smaller than other countries' data. So the two countries are easy to ensure economic growth. But for China, it must input

more part in total gross output. And the input proportion is becoming bigger and bigger from 1987 to 1992, and even to 2007.

IV. CONCLUSIONS

For a national economy considered as a linear system, it may be growing, decreasing, or keeping standstill. The input proportion in total gross output determines the economy status. For input coefficients matrix, its least eigenvalue in terms of absolute magnitude and largest eigenvalue are two critical limits. If the input proportion is less than the least eigenvalue, the economy must die out gradually.

Only when the input proportion is bigger than the largest eigenvalue, and the input structure brings into correspondence with the positive eigenvector, may the economy be growing.

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