© 2014 IJIRT | Volume 1 Issue 6 | ISSN : 2349-6002

STUDY OF FOURIER SERIES

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***Abstracts*- A graph of periodic function f(x) that has period L exhibits the same pattern every L units along the x-axis, so that f(x+L) = f(x) for every value of x. If we know what the function looks like over one complete period, we can thus sketch a graph of the function over a wider interval of x (that may contain many periods)**

I. FOURIER SERIES

Definition **1** (Periodic functions)
A function f(t) is said to have a period T or to be periodic with period T if for all t, f(t+T)=f(t), where T is a positive constant. The least value of T>0 is called the principal period or the fundamental period or simply the period of f(t).
Example

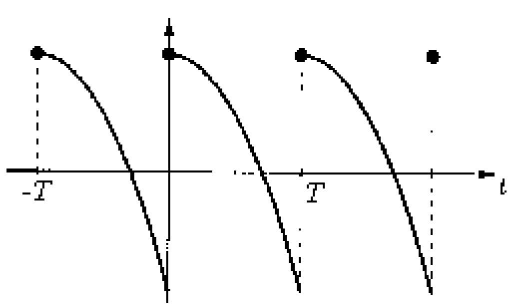
The function $\sin x$ has periods $2\pi, 4\pi, 6\pi, \ldots$, since

$\sin(x+2\pi), \sin(x+4\pi), \sin(x+6\pi), \ldots$

all equal $\sin x$.

Definition **2** (Periodic expansion)
Let a function f be declared on the interval [0,T). The periodic expansion $\tilde{f}$ of f is defined by the formula

$$\tilde{f}(t) = \begin{cases} f(t) & 0 \le t < T \\ \tilde{f}(t-T) & \forall t \in \mathbb{R} \end{cases}$$



Definition **3** (Piecewise continuous functions)
A function f defined on I=[a,b] is said to be piecewise continuous on I if and only if
(i) there is a subdivision $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ such that f is continuous on each subinterval $I_k = \{x : x_{k-1} < x < x_k\}$ and
(ii) at each of the subdivision points $x_0, x_1, \ldots, x_n$ both one-sided limits of f exist.

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\ nx + b_n \sin nx) \quad (1)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos ntdt \quad n = 0,1,2,\ldots$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin ntdt \quad n = 1,2,\ldots \quad (2)$$

Definition **4** (Fourier coefficients, Fourier series)
The numbers a_n and b_n are called the Fourier coefficients of f. When a_n and b_n are given by (2), the trigonometric series (1) is called the Fourier series of the function f.
Remark 1
If f is any integrable function then the coefficients a_n and b_n may be computed. However, there is no assurance that the Fourier series will converge to f if f is an arbitrary integrable function. In general, we write

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\ nx + b_n \sin nx)$$

to indicate that the series on the right may or may not converge to f at some points.

Remark 2 (Complex Notation for Fourier series) Using Euler's identities,

$$e^{i\theta} = \cos\theta + i\sin\theta$$

where i is the imaginary unit such that $i^2=-1$, the Fourier series of f(x) can be written in complex form as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \qquad (3)$$

where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \qquad (4)$$

and

$$c_0 = \frac{1}{2}a_0, \quad c_n = \frac{1}{2}(a_n - ib_n), \quad c_{-n} = \frac{1}{2}(a_n$$

$$a_0 = 2c_0, \quad a_n = c_n + c_{-n}, \quad b_n = i(c_n - c_{-n}$$

II. DIRICHLET CONDITIONS

It is important to establish simple criteria which determine when a Fourier series converges. In this section we will develop conditions on f(x) that enable us to determine the sum of the Fourier series. One quite useful method to analyse the convergence properties is to express the partial sums of a Fourier series as integrals. Riemann and Fejer have since provided other ways of summing Fourier series. In this section we limit the study of convergence to functions that are piecewise smooth on a given interval.

Definition **5** (Piecewise smooth function) A function f is piecewise smooth on an interval if both f and f' are piecewise continuous on the interval.

III. THE GIBBS PHENOMENON

Near a point, where f has a jump discontinuity, the partial sums S_n of a Fourier series exhibit a substantial overshoot near these endpoints, and an increase in n will not diminish the amplitude of the overshoot, although with increasing n the the overshoot occurs over smaller and smaller intervals. This phenomenon is called Gibbs phenomenon. In this section we examine some detail in the behaviour of the partial sums S_n of $S(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{k}$.

IV. FOURIER SERIES SUMMARY

- Any periodic function (period = L) can be written as a sum of sines and cosines, as:

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nx}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nx}{L}\right)}$$

The a_n and b_n numbers are called the "Fourier coefficients".

- a_0 represents the average value of the function, and is calculated by: $\boxed{a_0 = \frac{1}{L}\int_0^L f(x)dx}$
- The a_n coefficients can be calculated by the formula: $\boxed{a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{2\pi nx}{L}\right)dx}$
- The b_n coefficients can be calculated by the formula: $\boxed{b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{2\pi nx}{L}\right)dx}$
- If the function f(x) is even, only the cosine terms will be present. I.e., the b_n coefficients will all be zero.
- If the function f(x) is odd, only the sine terms will be present. I.e., the a_n coefficients will all be zero.
- All integrals can be done from –L/2 to L/2 instead of from 0 to L, if it makes things easier.

V. CONCLUSION

There are many different ways of defining Fourier series and Fourier coefficients, so you will often see slightly different equations compared to the four boxed ones above. My equations follow what I believe is the most common convention. Here are some specific things to watch out for, though:

- Dr. Durfee calls the cosine coefficient "b_n" and the sine coefficient "a_n", the reverse of what I said in the first bullet point above. However, I have NEVER seen anyone else do that (not textbooks, not professors, not websites), so I just couldn't bring myself to use Dr. Durfee's notation. NOTE: To avoid ambiguity, in the homework problems I refer to "the cosine coefficients" or "the sine coefficients" instead of to "a_n" and "b_n".
- Dr. Durfee uses the symbol "$□_0$" instead of L. I don't think that is a common symbol.
- Often the constant term is written as "$a_0/2$" instead of just "a_0". That makes a_0 be twice as big as in my definition above. They do that so that the a_n formula given in the third bullet point will also work for a_0.
- Sometimes people write the equations in terms of k instead of in terms of L, with $k = 2□/L$. They do that so that the arguments of the sines and cosines are simpler, just nkx. Often k defined that way is called k_0.
- If a Fourier series is written with time t as the variable instead of x, then typically T is used to represent the period instead of L, and $□_0$ is used instead of k_0. You can see that all terms besides the constant term are multiples of the lowest frequency, $□_0$.
- Sometimes people write the expansion in terms of exponentials instead of sines and cosines, using Euler's identity $e^{ix} = \cos x + i\sin x$ to combine the a_n and b_n coefficients into a single (complex) coefficient, typically called c_n.
- Sometimes L is defined to be 2□. In that case x must be rescaled so that it has the right period.

Regardless of the convention you use, when all is said and done the actual sines and cosines and their coefficients that make up the expansion (the first boxed equation above) have to be equivalent. For each function f(x), there is exactly one Fourier expansion that corresponds to it.

REFERENCES

[1] Lejeune-Dirichlet, P. "Sur la convergence des séries trigonométriques qui servent à représenter une fonction arbitraire entre des limites données". (In French), transl. "On the convergence of trigonometric series which serve to represent an arbitrary function between two given limits". Journal f¨ur die reine und angewandte Mathematik, Vol. 4 (1829) pp. 157–169.

[2] Ueber die Darstellbarkeit einer Function durch eine trigonometrische Reihe" [About the representability of a function by a trigonometric series]. *Habilitationschrift, Göttingen; 1854. Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen, vol. 13, 1867*. Published posthumously for Riemann by Richard Dedekind *(in German). Archived from the original on 20 May 2008. Retrieved 19 May 2008.*

[3] D. Mascre, Bernhard Riemann: Posthumous Thesis on the Representation of Functions by Triginometric Series (1867). Landmark Writings in Western Mathematics 1640–1940, Ivor Grattan-Guinness (ed.); pg. 492. Elsevier, 20 May 2005.Accessed 7 Dec 2012.</

[4] Theory of Complex Functions: Readings in Mathematics, by Reinhold Remmert; pg 29. Springer, 1991. Accessed 7 Dec 2012.

[5] Nerlove, Marc; Grether, David M.; Carvalho, Jose L. (1995). *Analysis of Economic Time Series. Economic Theory, Econometrics, and Mathematical Economics*. Elsevier. ISBN 0-12-515751-7.

[6] Flugge, Wilhelm (1957). *Statik und Dynamik der Schalen*. Berlin: Springer-Verlag.

[7] Dorf, Richard C.; Tallarida, Ronald J. (1993-07-15). *Pocket Book of Electrical Engineering Formulas* (1 ed.). Boca Raton,FL: CRC Press. pp. 171–174. ISBN 0849344735.

[8] Georgi P. Tolstov (1976). *Fourier Series*. Courier-Dover. ISBN 0-486-63317-9.

[9] "Gallica – Fourier, Jean-Baptiste-Joseph (1768–1830). Oeuvres de Fourier. 1888, pp. 218–219" (in French). Gallica.bnf.fr. 2007-10-15. Retrieved 2014-08-08.

[10] "fa.functional analysis - Characterizations of a linear subspace associated with Fourier

series". MathOverflow. 2010-11-19. Retrieved 2014-08-08.

[11] William McC. Siebert (1985). *Circuits, signals, and systems*. MIT Press. p. 402. ISBN 978-0-262-19229-3.

[12] L. Marton and Claire Marton (1990). *Advances in Electronics and Electron Physics*. Academic Press. p. 369. ISBN 978-0-12-014650-5.

[13] Hans Kuzmany (1998). *Solid-state spectroscopy*. Springer. p. 14. ISBN 978-3-540-63913-8.

[14] Karl H. Pribram, Kunio Yasue, and Mari Jibu (1991). *Brain and perception*. Lawrence Erlbaum Associates. p. 26. ISBN 978-0-89859-995-4.